

## Submanifolds of a nearly Kähler 6-dimensional sphere

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### 1. Introduction

It is well-known that a 6-dimensional unit sphere  $S^6$  admits a nearly Kähler structure  $(J, \langle \cdot, \cdot \rangle)$  and can be expressed by  $S^6 = G_2/SU(3)$  as a homogeneous almost Hermitian manifold, where  $G_2$  is the compact Lie group of all automorphisms of the octonions  $\mathfrak{C}$  (known also as the Cayley division algebra). This article is a brief survey on the several classes of submanifolds, that is,  $J$ -holomorphic curves (called also as almost complex submanifolds), totally real submanifolds and CR-submanifolds in  $S^6$ , and the intermediate relationships. In [3], Bolton, Vracken and Woodward have divided the  $J$ -holomorphic curves in 4 classes, I  $\sim$  IV. The class I is linearly full super-minimal  $J$ -holomorphic curves, the class II is linear full, but not super-minimal  $J$ -holomorphic curves, the class III is linearly full in some totally geodesic  $S^5$  in  $S^6$  (then necessarily not super-minimal) and the class IV is totally geodesic  $J$ -holomorphic curves. We note that any example of the class II has been not obtained up to our knowledge. §3 will be devoted to a brief survey on  $J$ -holomorphic curves in  $S^6$ . In §4 we discuss 3-dimensional totally real submanifolds and CR-submanifolds in  $S^6$ , and further their intermediate relationships. In §5, we discuss 4-dimensional CR-submanifolds in  $S^6$ . In particular, we introduce some interesting examples of such submanifolds. The present article is much indebted to ([19]).

### 2. Preliminaries

First, we shall recall the definition of the almost complex structure of the 6-dimensional sphere. To do this, we recall the fundamental relations between

the quaternions and the octonions (or called as the Cayley algebra). Let  $\mathbf{H} = \text{span}_{\mathbf{R}}\{1, i, j, k\}$  be the quaternions with its canonical basis  $1, i, j, k$  which satisfy

$$i^2 = j^2 = k^2 = -1,$$

$$ij = -ji = k, \quad ki = -ik = j, \quad jk = -kj = i.$$

Next, let  $\mathfrak{C}$  be the octonions and  $\text{Im}\mathfrak{C}$  be the its pure imaginary part, respectively. The octonions  $\mathfrak{C}$  can be identified with the direct sum of the two quaternions  $\mathbf{H} \oplus \mathbf{H}$  with the following multiplication. We represent  $(a, b) \in \mathbf{H} \oplus \mathbf{H}$  as  $a + b\varepsilon$ .

$$(a + b\varepsilon)(c + d\varepsilon) = (ac - \bar{d}b) + (da + b\bar{c})\varepsilon.$$

where  $a, b, c, d$  are quaternions and  $\varepsilon = (0, 1) \in \mathbf{H} \oplus \mathbf{H}$ , and  $\bar{a}$  is the conjugate of  $a$ . With respect to this multiplication, the octonions  $\mathfrak{C}$  is an non-commutative, non-associative, alternative (the multiplication satisfy  $(xy)y = xy^2, (xy)x = x(yx), y(yx) = y^2x$ ) division algebra. We denote  $\langle \cdot, \cdot \rangle$  the canonical inner product on 8-dimensional Euclidean space. Then the multiplication of the octonions and the inner product satisfy the following (which is called the normed algebra by Harvey-Lawson [15])

$$\langle xy, xy \rangle = \langle x, x \rangle \langle y, y \rangle$$

where  $x, y$  are octonions. We can define the exterior product  $x \times y$  by  $x \times y = (1/2)(\bar{y}x - \bar{x}y) \in \text{Im}\mathfrak{C}$ . where  $\bar{x} = 2 \langle x, 1 \rangle - x$  is the conjugate of  $x$  in  $\mathfrak{C}$ . If  $x \in \text{Im}\mathfrak{C}$  then we have  $\bar{x} = -x$ . More we assume that  $x, y \in \text{Im}\mathfrak{C}$  and  $\langle x, y \rangle = 0$ , then  $x \times y = xy$ .

We define the tensor field  $J$  of type (1,1), on the 6-dimensional sphere  $S^6 = \{x \in \text{Im}\mathfrak{C} \mid \langle x, x \rangle = 1\}$  by;

$$J_x X = X \times x$$

for any  $X \in T_x S^6, x \in S^6$ . Then we see that  $J_x^2 = -I$  and  $J$  is orthogonal with respect to the induced metric. We define the section  $J$  of the bundle of the endomorphisms  $\text{End}(TS^6)$  of the tangent bundle  $TS^6$ , such that  $J(x) = J_x$ . Then  $(S^6, J, \langle \cdot, \cdot \rangle)$  is an almost Hermitian manifold. The group of automorphism of the almost Hermitian structure  $\text{Aut}(S^6, J, \langle \cdot, \cdot \rangle) = \{f \mid f : S^6 \rightarrow S^6 \text{ isometry with } f_* \circ J = J \circ f_*\}$  coincides with  $G_2 = \{g \in SO(7) \mid g(uv) = g(u)g(v) \text{ for any } u, v \in \mathfrak{C}\}$ .

Next, we shall give the multiplication table of the octonions. We set the basis of the octonions as follows;  $e_0 = (1, 0), e_1 = (i, 0), e_2 = (j, 0), e_3 = (k, 0), e_4 = (0, 1) = \varepsilon, e_5 = (0, i) = i\varepsilon, e_6 = (0, j) = j\varepsilon, e_7 = (0, k) = k\varepsilon$ . Then we have The element of the exceptional Lie group  $G_2$  is obtained by the following manner, which may be compared with orthonormalization process by Gram-Schmidt.

$i \setminus j$	1	2	3	4	5	6	7
1	$-e_0$	$e_3$	$-e_2$	$e_5$	$-e_4$	$-e_7$	$e_6$
2	$-e_3$	$-e_0$	$e_1$	$e_6$	$e_7$	$-e_4$	$-e_5$
3	$e_2$	$-e_1$	$-e_0$	$e_7$	$-e_6$	$e_5$	$-e_4$
4	$-e_5$	$-e_6$	$-e_7$	$-e_0$	$e_1$	$e_2$	$e_3$
5	$e_4$	$-e_7$	$e_6$	$-e_1$	$-e_0$	$-e_3$	$e_2$
6	$e_7$	$e_4$	$-e_5$	$-e_2$	$e_3$	$-e_0$	$-e_1$
7	$-e_6$	$e_5$	$e_4$	$-e_3$	$-e_2$	$e_1$	$-e_0$

**Lemma 2.1.** *Let  $a_1, a_2$  be an orthonormal pair of  $\text{Im}\mathfrak{C}$ , and put  $a_3 = a_1 a_2$ . We set a unit vector  $a_4$  of  $\text{Im}\mathfrak{C}$  which is orthogonal to the associative plane spanned by  $\{a_1, a_2, a_3\}$ . Next we put  $a_5 = a_1 \cdot a_4$ ,  $a_6 = a_2 \cdot a_4$ ,  $a_7 = a_3 \cdot a_4$ . Then*

$$g = (a_1, a_2, a_3, a_4, a_5, a_6, a_7) \in SO(7)$$

*is an element of  $G_2$  and satisfy  $g \cdot e_4 = a_4$ .*

In fact, we can easily see that the multiplication table of  $\{a_1, a_2, a_3, a_4, a_5, a_6, a_7\}$  can be obtained by replacing  $e_i$  by  $a_i$  in the above multiplication table.

Next, we shall give the Lie algebra  $\mathfrak{g}_2$  of  $G_2$  as a Lie subalgebra of  $\mathfrak{so}(7)$ . To do this, we take the basis  $G_{ij}$  ( $1 \leq i \neq j \leq 7$ ) of  $\mathfrak{so}(7)$  which is defined by

$$G_{ij}(e_k) = \begin{cases} e_i, & \text{if } k = j, \\ -e_j, & \text{if } k = i, \\ 0, & \text{otherwise.} \end{cases}$$

Then the Lie algebra  $\mathfrak{g}_2$  coincide with the subspace of  $\mathfrak{so}(7)$  which is spanned by

$$(2.1) \quad \begin{cases} aG_{23} + bG_{45} + cG_{76}, \\ aG_{31} + bG_{46} + cG_{57}, \\ aG_{12} + bG_{47} + cG_{65}, \\ aG_{51} + bG_{73} + cG_{62}, \\ aG_{14} + bG_{72} + cG_{36}, \\ aG_{17} + bG_{24} + cG_{53}, \\ aG_{61} + bG_{34} + cG_{25}, \end{cases}$$

where  $a, b, c \in \mathbf{R}$  satisfy  $a+b+c = 0$ . By Lemma 2.1, we see that  $G_2$  acts transitively on  $S^6$ . Hence, for any point  $x \in S^6$ , there is a  $g \in G_2$  such that  $x = g(\varepsilon)$ , and we have  $J_x = g \circ J_\varepsilon \circ g^{-1}$ . We see that  $(S^6, J, \langle, \rangle)$  is a homogeneous almost

Hermitian manifold. We denote by  $D$  the Levi-Civita connection of  $S^6$ . Then we have

$$(D_X J)Y = -X \times Y + \langle X \times Y, x \rangle x,$$

for  $X, Y \in T_x S^6, x \in S^6$ . Thus we see that the almost Hermitian structure  $(J, \langle, \rangle)$  on  $S^6$  is a nearly Kähler structure  $((D_X J)X = 0)$  which is not Kähler one.

Now, we prepare fundamental formula for Riemannian submanifolds of  $S^6$ . Let  $(M, \phi)$  be a submanifolds of  $S^6$  with the isometric immersion  $\phi : M \rightarrow S^6$ . We set  $x = \iota \circ \phi$  and consider  $x$  as the corresponding position vector to the image of  $\phi$  in  $\text{Im}\mathfrak{C}$ , where  $\iota$  denote the inclusion map from  $S^6$  to  $\text{Im}\mathfrak{C}$ . We denote by  $\nabla$  and  $\nabla^\perp$  the Riemannian connections on  $M$  and the normal bundle  $T^\perp M$  induced by the Riemannian connection  $D$  on  $S^6$ , respectively. Then, the Gauss and Weingarten formulas are given respectively by

$$(2.2) \quad D_X Y = \nabla_X Y + \sigma(X, Y),$$

$$(2.3) \quad D_X \xi = -A_\xi X + \nabla_X^\perp \xi,$$

where  $\sigma$  and  $A_\xi$  are the second fundamental form and the shape operator (with respect to the normal vector field  $\xi$ ) respectively, and  $X, Y \in \mathfrak{X}(M)$  where  $\mathfrak{X}(M)$  denotes the Lie algebra of all smooth tangent vector fields on  $M$ . The second fundamental form  $\sigma$  and the shape operator  $A_\xi$  are related by

$$\langle \sigma(X, Y), \xi \rangle = \langle A_\xi X, Y \rangle.$$

The Gauss, Codazzi and Ricci equations are given respectively by

$$(2.4) \quad \begin{aligned} \langle R(X, Y)Z, W \rangle &= \langle X, W \rangle \langle Y, Z \rangle - \langle X, Z \rangle \langle Y, W \rangle \\ &+ \langle \sigma(X, W), \sigma(Y, Z) \rangle - \langle \sigma(X, Z), \sigma(Y, W) \rangle \end{aligned}$$

$$(2.5) \quad (\nabla'_X \sigma)(Y, Z) = (\nabla'_Y \sigma)(X, Z),$$

$$(2.6) \quad \langle R^\perp(X, Y)\xi, \eta \rangle = \langle [A_\xi, A_\eta]X, Y \rangle,$$

where  $(\nabla'_X \sigma)(Y, Z) = \nabla_X^\perp \sigma(Y, Z) - \sigma(\nabla_X Y, Z) - \sigma(Y, \nabla_X Z)$ , and  $R^\perp(X, Y)\xi = [\nabla_X^\perp, \nabla_Y^\perp]\xi - \nabla_{[X, Y]}^\perp \xi$  for  $X, Y, Z, W \in \mathfrak{X}(M)$  and  $\xi, \eta$  are vector fields normal to  $M$ .

We herewith recall the definitions of almost complex submanifolds, totally real submanifolds and CR-submanifolds of a nearly Kähler  $S^6$ . A submanifold  $(M, x)$  of  $S^6$  is called an almost complex submanifolds (or invariant submanifolds) of  $S^6$  if the tangent bundle  $TM$  of  $M$  is stable under the action of  $J$ , that is,  $x_*(J(TM)) = x_*(TM)$ . It is well-known that there does not exist a 4-dimensional almost complex submanifold of  $S^6$  ([13]). However, it is also known there are many examples of 2-dimensional almost complex submanifolds of  $S^6$ . We shall call a 2-dimensional almost complex submanifold a J-holomorphic curve in the present article. A submanifold  $(M, x)$  is called a totally real submanifold of  $S^6$  if the image of  $J$  of the tangent bundle  $TM$  of  $M$  is contained in the normal bundle  $T^\perp M$ , that is,  $x_*(J(TM)) \subset T^\perp M$ . In this case, we easily see that  $\dim M \leq 3$ . Further, a submanifold  $(M, x)$  is called a CR-submanifold of  $S^6$  if the tangent bundle  $TM$

of  $M$  admits a splitting  $TM = H \oplus H^\perp$  such that  $H$  is  $J$ -invariant and  $H^\perp$  is totally real, that is  $x_*(H^\perp) \subset T^\perp M$ . If  $H$  and  $H^\perp$  are both non-trivial, the CR-submanifold  $(M, x)$  is said to be proper. In this article, we consider only proper ones as CR-submanifolds of  $S^6$ .

### 2.1. Structure equation of $G_2$

We recall the structure equation of  $G_2$  obtained by Bryant ([5]). We denote by  $M_{p \times q}(\mathbf{C})$  the set of  $p \times q$  complex matrices and  $[a] \in M_{3 \times 3}(\mathbf{C})$  is given by (this matrix corresponds to the usual exterior product of  $R^3$ )

$$[a] = \begin{pmatrix} 0 & a_3 & -a_2 \\ -a_3 & 0 & a_1 \\ a_2 & -a_1 & 0 \end{pmatrix}$$

where  $a = {}^t(a_1 \ a_2 \ a_3) \in M_{3 \times 1}(\mathbf{C})$ . Then we have

$$[a]b + [b]a = 0$$

where  $a, b \in M_{3 \times 1}(\mathbf{C})$ . Now, we set a basis of  $\mathbf{C} \otimes_R \text{Im} \mathfrak{C}$  with respect to  $J_\varepsilon$ .

$$\varepsilon = (0, 1) \in \mathbf{H} \oplus \mathbf{H},$$

$$\mathbf{E}_1 = i\mathbf{N}, \quad \mathbf{E}_2 = j\mathbf{N}, \quad \mathbf{E}_3 = -k\mathbf{N},$$

$$\bar{\mathbf{E}}_1 = i\bar{\mathbf{N}}, \quad \bar{\mathbf{E}}_2 = j\bar{\mathbf{N}}, \quad \bar{\mathbf{E}}_3 = -k\bar{\mathbf{N}},$$

where  $\mathbf{N} = (1 - \sqrt{-1}\varepsilon)/2$ ,  $\bar{\mathbf{N}} = (1 + \sqrt{-1}\varepsilon)/2 \in \mathbf{C} \otimes_R \mathfrak{C}$ . The basis  $(u, f, \bar{f})$  of  $\mathbf{C} \otimes_R \text{Im} \mathfrak{C}$  is called a  $G_2$ -admissible frame if there is  $g \in G_2 \subset M_{7 \times 7}(\mathbf{C})$  such that

$$(u, f, \bar{f}) = (g(\varepsilon), g(\mathbf{E}), g(\bar{\mathbf{E}})) = (\varepsilon, \mathbf{E}, \bar{\mathbf{E}})g.$$

We may identify a  $G_2$ -admissible frame with an element of  $G_2$ . Then the structure equations of  $G_2$  are given by the following;

**Proposition 2.1.** ([5]) *There exist left invariant 1-forms  $\kappa$  and  $\theta$  on  $G_2$ . Then*

$$\begin{aligned} d(u, f, \bar{f}) &= (u, f, \bar{f}) \begin{pmatrix} 0 & -\sqrt{-1} {}^t \bar{\theta} & \sqrt{-1} {}^t \theta \\ -2\sqrt{-1} \theta & \kappa & [\bar{\theta}] \\ 2\sqrt{-1} \bar{\theta} & [\theta] & \bar{\kappa} \end{pmatrix} \\ &= (u, f, \bar{f}) \Phi, \end{aligned}$$

where  $\theta = (\theta^i)$  is a  $M_{3 \times 1}(\mathbf{C})$ -valued 1-form,  $\kappa = (\kappa_j^i)$ ,  $1 \leq i, j \leq 3$ , is a  $\mathfrak{su}(3)$ -valued 1-form. We denote by  $\Phi$  a Maurer-Cartan form of  $G_2$ . Therefore  $\Phi$  satisfy the integrability condition  $d\Phi = -\Phi \wedge \Phi$ . Or equivalently, we have

$$d\theta = -\kappa \wedge \theta + [\bar{\theta}] \wedge \bar{\theta}.$$

$$d\kappa = -\kappa \wedge \kappa + 3\theta \wedge {}^t\bar{\theta} - ({}^t\theta \wedge \bar{\theta}) I_3.$$

We can easily see that  $\kappa$  defines the  $SU(3)$ -connection on  $S^6$ . Also,  $d\kappa + \kappa \wedge \kappa$  generates the Lie algebra of  $SU(3)$ -holonomy group.

### 3. J-holomorphic curves

#### 3.1. Fundamental properties

Let  $M$  be an oriented real 2-dimensional surface and  $\phi : M \rightarrow S^6$  be a J-holomorphic curve of  $S^6$ . Then, we have the following property with respect to the isothermal coordinate  $z$  which compatible with the oietation,

$$J\phi_* \left( \frac{\partial}{\partial z} \right) = \sqrt{-1} \phi_* \left( \frac{\partial}{\partial z} \right)$$

**Proposition 3.1.** *Let  $\phi : M \rightarrow S^6$  be a J-holomorphic curve of  $S^6$ . Then we have*

(1)

$$\sigma(JX, Y) = \sigma(X, JY) = J\sigma(X, Y).$$

*In particular, any J-holomorphic curve is a minimal submanifold of  $S^6$ .*

(2)

$$A_{J\xi}(X) = J(A_\xi X).$$

(3)

$$\nabla_X^\perp(J\xi) = J(\nabla_X^\perp \xi) + \xi \times X.$$

where  $\sigma$  is the second fundamental form of  $x$ ,  $A_\xi$  is a shape operator with respect to the normal vector  $\xi$  and  $\nabla^\perp$  is an induced connection of the normal bundle, respectively, and  $X, Y$  are vector fields on  $M$ .

#### 3.2. $G_2$ -frame fields

In this section, we introduce the  $G_2$ -frame field along the J-holomorphic curve of  $S^6$ , and give the structure equations with respect to the  $G_2$  frame field. First we note that  $G_2$  can be regarded as the total space of the principal  $SU(3)$ -right bundle over  $S^6$ . We assume there exists the small open subset  $U \subset M$  such that any point of  $U$  is not geodesic one. Then we shall construct the  $SU(3)$ -frame field on  $U$  as follows. Let  $\{e_1, Je_1\}$  be a local tangential orthonormal frame field on  $U$ ,  $\{\xi_1, J\xi_1\}$  and  $\{e_1 \times \xi_1, J(e_1 \times \xi_1)\}$  the the local orthonormal frame field of the 1-st normal

bundle and the 2-nd normal bundle, respectively. We take the complexification of these frame fields, we define the  $SU(3)$  frame field  $(f, \bar{f})$  on  $x(U) \subset S^6$  in  $\mathbf{C} \otimes_{\mathbf{R}} \text{Im} \mathcal{C}$

$$(3.1) \quad \begin{aligned} f_3 &= (1/2)(e_1 - \sqrt{-1}J e_1), \\ f_2 &= (1/2)(\xi_1 - \sqrt{-1}J \xi_1), \\ f_1 &= (1/2)(e_1 \times \xi_1 - \sqrt{-1}J(e_1 \times \xi_1)), \end{aligned}$$

Then,  $(x, f_1, f_2, f_3, \bar{f}_1, \bar{f}_2, \bar{f}_3) = (x, f, \bar{f})$  is the local admissible  $G_2$ -frame field on  $U$  along the immersion  $x$ . Therefore, there exists a  $g \in G_2$  such that

$$\begin{aligned} f_3 &= (1/2)(g(-k) - \sqrt{-1}g(-k\varepsilon)) = g(\mathbf{E}_3), \\ f_2 &= (1/2)(g(j) - \sqrt{-1}g(j\varepsilon)) = g(\mathbf{E}_2), \\ f_1 &= (1/2)(g(i) - \sqrt{-1}g(i\varepsilon)) = g(\mathbf{E}_1). \end{aligned}$$

Pulling back the  $G_2$ -structure equation by the immersion  $x$ , we get the corresponding structure equation in the following Proposition along the curve  $x$ . The frame field  $(x, f, \bar{f})$  belongs to the sub-bundle of the induced  $G_2$ -bundle.

**Proposition 3.2.** *Let  $x : M \rightarrow S^6$  be a  $J$ -holomorphic curve of the 6-dimensional sphere  $S^6$ . Then we have*

$$\begin{aligned} dx &= f_3(-2\sqrt{-1}\theta^3) + \bar{f}_3(2\sqrt{-1}\bar{\theta}^3), \\ \theta^2 &= \theta^1 = 0, \quad \kappa_3^1 = 0, \\ df_3 &= x(-\sqrt{-1}\bar{\theta}^3) + \sum_{i=1}^3 f_i \cdot \kappa_3^i \quad (\text{Gauss formula}), \\ df_2 &= \sum_{i=1}^3 f_i \cdot \kappa_2^i + \bar{f}_1\theta^3, \\ df_1 &= \sum_{i=1}^3 f_i \cdot \kappa_1^i - \bar{f}_2\theta^3, \quad (\text{Weingarten formula}). \end{aligned}$$

Since the isotropy group is  $SU(3)$ , we get

$$\kappa_3^3 + \kappa_2^2 + \kappa_1^1 = 0,$$

The integrability conditions imply that

$$\begin{aligned} d\theta^3 + \kappa_3^3 \wedge \theta^3 &= 0, \\ d\kappa_3^3 + \kappa_2^3 \wedge \kappa_3^2 &= 2\theta^3 \wedge \bar{\theta}^3, \\ d\kappa_2^2 + \kappa_3^2 \wedge \kappa_2^3 + \kappa_1^2 \wedge \kappa_2^1 &= -\theta^3 \wedge \bar{\theta}^3, \\ d\kappa_1^1 + \kappa_2^1 \wedge \kappa_1^2 &= -\theta^3 \wedge \bar{\theta}^3, \\ d\kappa_3^2 + (\kappa_2^2 - \kappa_3^3) \wedge \kappa_3^2 &= 0, \\ d\kappa_2^1 + (\kappa_1^1 - \kappa_2^2) \wedge \kappa_2^1 &= 0. \end{aligned}$$

We note that  $\kappa_3^2, \kappa_2^1$  are local 1-form of type  $(1, 0)$  with respect to the induced complex structure (from the induced metric) of  $M$ .

**Remark 3.1.** The immersion  $x$  is not totally geodesic, we see that the geodesic points are isolated (if exist) (refer to the argument in §3.4). In this case the  $SU(3)$ -frame field given in (3.1) can be extended to a neighborhood containing the geodesic point by making use of  $SU(3)$ -connection.

### 3.3. Examples of constant Gauss curvature

The second author ([32]) proved that if  $M$  is a J-holomorphic curve of constant curvature  $K$  in  $S^6$ , then  $K = 1, 1/6$  or  $0$ . We may see that such surfaces are congruent up to the action of  $SO(7)$  to the following three examples.

(1) Totally geodesic J-holomorphic sphere.

$$\iota : V \cap S^6 \rightarrow S^6$$

where  $V$  is an associative plane and  $\iota$  is an inclusion map.

(2) Boruvka sphere;

$$\varphi : S^2(1/6) \rightarrow S^6$$

where  $\varphi$  is defined by

$$\begin{aligned} \varphi(y_1, y_2, y_3) &= (1/24\sqrt{6}) \{i(-\sqrt{10})y_2(y_2^2 - 3y_3^2) \\ &+ j(-2\sqrt{15})y_1(y_2^2 - y_3^2) \\ &+ k(\sqrt{6})y_3(y_2^2 + y_3^2 - 4y_1^2) \\ &+ \varepsilon 2y_1(3y_2^2 + 3y_3^2 - 2y_1^2) \\ &+ i\varepsilon(\sqrt{10})y_3(3y_2^2 - y_3^2) \\ &+ j\varepsilon(4\sqrt{15})(y_1y_2y_3) \\ &+ k\varepsilon(\sqrt{6})y_2(y_2^2 + y_3^2 - 4y_1^2)\} \end{aligned}$$

where  $y_1^2 + y_2^2 + y_3^2 = 6$ , and its Gauss curvature is identically  $1/6$ . The mapping  $\varphi$  is an imbedding.

(3). Flat torus:

$$f : T^2 \rightarrow S^6$$

where  $f$  is defined by

$$\begin{aligned} f(z, \bar{z}) &= (1/\sqrt{3})\{E_1 e^{z-\bar{z}} + \overline{E_1} e^{-(z-\bar{z})} \\ &+ E_2 e^{\omega z - \bar{\omega} \bar{z}} + \overline{E_2} e^{-(\omega z - \bar{\omega} \bar{z})} \\ &+ E_3 e^{\omega^2 z - \bar{\omega}^2 \bar{z}} + \overline{E_3} e^{-(\omega^2 z - \bar{\omega}^2 \bar{z})}\} \\ &= (1/\sqrt{3})\{i \cos(2y) + i\varepsilon \sin(2y) \\ &+ j \cos(\sqrt{3}x - y) + j\varepsilon \sin(\sqrt{3}x - y) \\ &- k \cos(\sqrt{3}x + y) + k\varepsilon \sin(\sqrt{3}x + y)\} \end{aligned}$$

where  $\omega = e^{2\pi\sqrt{-1}/3}$  and  $z = x + \sqrt{-1}y \in \mathbf{C}$ . We may remark that the flat torus is not  $G_2$ -congruent in general.

**Remark 3.2.** R.L.Bryant ([6]) classified the minimal surfaces of constant Gauss curvature of space forms.

### 3.4. Local existence and the $G_2$ -congruency

First, we introduce the notion of some class of functions on Riemann surface  $M$  with holomorphic coordinate  $z$ . A non-negative function  $f : M \rightarrow \mathbf{R}_{\geq 0}$  is called an absolute value type, if for each point of  $M$  there exists some neighborhood  $U$  and the holomorphic function  $\varphi_0(z)$  and non-zero complex valued function  $\varphi_1(z, \bar{z})$  such that  $f(z, \bar{z}) = |\varphi_0(z)\varphi_1(z, \bar{z})|$  on  $U$ .

By Proposition 3.2, we have

**Theorem 3.1.** ([16]) *Let  $M$  be a simply connected surface. If its Gauss curvature  $K$  is not identically 1, and satisfy  $K \leq 1$ . We also assume that the functions  $\sqrt{1-K}$  and  $|\mathbf{III}|$  are of absolute value type, and further following differential equalities*

$$\Delta \log(1 - K) = 6K - 1 + 4 |\mathbf{III}|^2,$$

$$\Delta \log |\mathbf{III}| = 1 - 4 |\mathbf{III}|^2,$$

hold outside their zeros. Then there exists a  $J$ -holomorphic curves  $\varphi : M \rightarrow S^6$  where  $|\mathbf{III}| = |\kappa_2^1(f_3)|$ .

Let  $x : M \rightarrow S^6$  be a  $J$ -holomorphic curve of  $S^6$ . We set  $\Lambda$  as

$$\Lambda = 4\sqrt{-1}(\kappa_3^2)^2 \otimes \kappa_2^1 \otimes (\theta^3)^3,$$

then it is a holomorphic 6-differential on  $M$ . If we take the local isothermal coordinate  $z$ , then we have

$$\Lambda = \langle \sigma_2(\partial_z, \partial_z, \partial_z), x_*(\partial_z) \times \sigma(\partial_z, \partial_z) \rangle dz^6.$$

where  $\partial_z = \partial/\partial z$ , and  $\sigma_2$  is a third fundamental form. By Theorem 3.1, we can obtain the classification theorems of  $J$ -holomorphic curves with constant Gauss curvature (as above in §3.3).

**Theorem 3.2.** ([16]) *Let  $M$  be a connected, orientable surface and  $x_1, x_2 : M \rightarrow S^6$  be two  $J$ -holomorphic curves. Then there exists an element  $g \in G_2$  such that  $g \circ x_1 = x_2$  if and only if the induced metrics and the corresponding holomorphic 6-differentials coincide.*

The J-holomorphic curve is said to be *super-minimal* if the holomorphic 6-differential identically zero.

### 3.5. Weierstraß-Bryant type formula for super-minimal J-holomorphic curves

Let  $Q^5$  be the 5-dimensional complex quadric  $Q^5 \subset P^6(\mathbf{C})$  of the complex projective space, given by the following

$$Q^5 = \{[w_0 : w_1 : \cdots : w_6] \in P^6(\mathbf{C}) \mid (w_0)^2 + w_1 w_4 + w_2 w_5 + w_3 w_6 = 0\}$$

where  $[ \ ]$  is the homogeneous coordinate of  $P^6(\mathbf{C})$ . Any super-minimal J-holomorphic curves of  $S^6$  can be obtained as the projection of the integral curve of the (complex) 2-dimensional sub-bundle  $L_+$  (with respect to the  $SU(3)$ -connection), of the holomorphic tangent bundle  $T^{(1,0)}Q^5$  of the quadrics  $Q^5$ . The differential equations which is corresponding to the integral curve, can be rewritten as the ordinary differential equation with respect to the  $G_2(\mathbf{C})$ -valued function. By solving them, we have the following representation formula.

**Theorem 3.3.** ([18]) *Let  $U$  be a simply connected domain of  $\mathbf{C}$ . The map  $\Xi : U \rightarrow Q^5 \subset P^6(\mathbf{C})$  defined by the following, is an integral curve of  $L_+$  with initial condition  $\Xi(0) = E_1$ . where  $f(\zeta)$  is a holomorphic function on  $U$ .*

$$\Xi(\zeta) = \varepsilon \alpha_1(\zeta) + \mathbf{E}_1 \cdot 1 + \mathbf{E}_2 \alpha_2(\zeta) + \mathbf{E}_3 \alpha_3(\zeta) + \bar{\mathbf{E}}_1 \alpha_4(\zeta) + \bar{\mathbf{E}}_2 \alpha_5(\zeta) + \bar{\mathbf{E}}_3 \zeta$$

where

$$\begin{aligned} \alpha_1(\zeta) &= (\sqrt{-1}/2) \left[ \zeta \left( f''(\zeta) + f''(0) \right) - 2 \left( f'(\zeta) - f'(0) \right) \right], \\ \alpha_2(\zeta) &= (1/2) \zeta^2 f''(\zeta) - \zeta \left( 2f'(\zeta) + f'(0) \right) + 3 \left( f(\zeta) - f(0) \right), \\ \alpha_3(\zeta) &= (1/2) f''(\zeta) \left( f'(0) - f'(\zeta) \right) \\ &\quad + f''(0) \left[ (1/2) (\zeta f''(\zeta)) - f'(\zeta) + f'(0) + (1/4) (f''(0) \zeta) \right] \\ &\quad + (3/4) \int_0^\zeta (f''(z))^2 dz, \\ \alpha_4(\zeta) &= f''(\zeta) \left[ -(3/2) f(\zeta) + (1/2) \zeta f'(\zeta) + f'(0) \zeta + (1/4) (f''(0) \zeta^2) + (3/2) f(0) \right] \\ &\quad + f'(\zeta) \left( f'(\zeta) - f''(0) \zeta - 2f'(0) \right) + (3/2) f''(0) f(\zeta) \\ &\quad - (3/4) \zeta \int_0^\zeta (f''(z))^2 dz \\ &\quad - (1/2) (f'(0) f''(0) \zeta) - (3/2) f(0) f''(0) + (f'(0))^2, \\ \alpha_5(\zeta) &= (1/2) (f''(\zeta) - f''(0)). \end{aligned}$$

As an application, we get the following 1-parameter family of the integral curves  $\Xi_t : P^1(\mathbf{C}) \rightarrow Q^5$  of  $L_+$ .

$$\Xi_t(z) = (\varepsilon, \mathbf{E}, \overline{\mathbf{E}}) \begin{pmatrix} \iota \sqrt{10t} e^{i\theta} z^3 \\ 1 \\ \sqrt{15t} e^{i\theta} z^4 \\ -\sqrt{6} t e^{2i\theta} z^5 \\ t e^{2i\theta} z^6 \\ \sqrt{15t} e^{i\theta} z^2 \\ \sqrt{6} z \end{pmatrix},$$

for  $z \in \mathbf{C}$ . Since the projection from  $S^6$  to  $Q^5$  is given by

$$\mathbf{x}_t(z) = \frac{\iota \Xi_t(z) \times \overline{\Xi_t(z)}}{\langle \Xi_t(z), \overline{\Xi_t(z)} \rangle},$$

the corresponding super-minimal J-holomorphic curves of  $S^6$  are given by

$$\frac{1}{\langle \Xi_t(z), \overline{\Xi_t(z)} \rangle} (\varepsilon, \mathbf{E}, \overline{\mathbf{E}}) \begin{pmatrix} (1/2)(1 - 6p - 15tp^2 + 15tp^4 + 6t^2p^5 - t^2p^6) \\ \iota e^{-i\theta} \sqrt{10t} \bar{z}^3 (1 + 3p + 3tp^2 + tp^3) \\ -\iota \sqrt{6} z (1 - 5tp^2 - 5tp^3 + t^2p^5) \\ \iota \sqrt{15t} e^{i\theta} z^2 (1 + 2p - 2tp^3 - tp^4) \\ -\iota e^{i\theta} \sqrt{10t} z^3 (1 + 3p + 3tp^2 + tp^3) \\ \iota \sqrt{6} \bar{z} (1 - 5tp^2 - 5tp^3 + t^2p^5) \\ -\iota \sqrt{15t} e^{-i\theta} \bar{z}^2 (1 + 2p - 2tp^3 - tp^4) \end{pmatrix}$$

where  $p = z\bar{z} = |z|^2$  and

$$\langle \Xi_t(z), \overline{\Xi_t(z)} \rangle = (1/2)(1 + 6p + 15tp^2 + 20tp^3 + 15tp^4 + 6t^2p^5 + t^2p^6) \geq 1/2.$$

From this formula, we can calculate the Gauss curvature of the above super-minimal J-holomorphic curves of  $S^6$ , and by Theorem 3.2, we get the following.

**Theorem 3.4.** ([18]) *The 1-parameter family of the above super-minimal J-holomorphic curves of  $S^6$ ,  $\mathbf{x}_t : P^1(\mathbf{C}) \rightarrow S^6$  ( $0 \leq t \leq 1$ ) gives a (non-trivial) deformation which are not  $G_2$ -congruent. In particular, if  $t = 1$ , then it is a Boruvka sphere, and if  $t = 0$  then it is totally geodesic.*

### 3.6. The cohomogeneity one J-holomorphic curves of $S^5$

In [23], they discussed the construction and the classification of compact J-holomorphic curves of class III, which are homeomorphic to torus. In this section,

we shall write the system of ordinary differential equations of a J-holomorphic curves of class III with  $S^1$ -symmetry.

First, we consider the map  $\phi : \mathbf{R}^2 \rightarrow S^5$  defined by

$$(3.2) \quad \phi(x, y) = (E, \bar{E}) \begin{pmatrix} e^{\sqrt{-1}\alpha_1 y} p_1(x) \\ e^{\sqrt{-1}\alpha_2 y} p_2(x) \\ e^{\sqrt{-1}\alpha_3 y} p_3(x) \\ e^{-\sqrt{-1}\alpha_1 y} \overline{p_1(x)} \\ e^{-\sqrt{-1}\alpha_2 y} \overline{p_2(x)} \\ e^{-\sqrt{-1}\alpha_3 y} \overline{p_3(x)} \end{pmatrix} = \begin{pmatrix} e^{\sqrt{-1}\alpha_i y} p_i(x) \\ e^{-\sqrt{-1}\alpha_i y} \overline{p_i(x)} \end{pmatrix},$$

where  $\alpha_i \in \mathbf{R}$ ,  $\alpha_1 + \alpha_2 + \alpha_3 = 0$  and  $\alpha_1 \alpha_2 \alpha_3 \neq 0$  and  $p_i(x)$  ( $i = 1, 2, 3$ ) are some complex-valued smooth functions of  $x \in \mathbf{R}$  with  $\sum_{i=1}^3 |p_i(x)|^2 = 1$ . We may easily observe that  $\phi$  is an immersion and the image  $\phi(\mathbf{R}^2) \subset S^5$  (totally geodesic in  $S^6$ ), if the conditions (3.6)  $\sim$  (3.8) (see below) are satisfied. The tangent space of the submanifold  $(\mathbf{R}^2, \phi)$  is spanned by

$$(3.3) \quad \phi_*(\partial/\partial x) = (E, \bar{E}) \begin{pmatrix} e^{\sqrt{-1}\alpha_i y} p'_i(x) \\ e^{-\sqrt{-1}\alpha_i y} \overline{p'_i(x)} \end{pmatrix}$$

$$(3.4) \quad \phi_*(\partial/\partial y) = (E, \bar{E}) \begin{pmatrix} \sqrt{-1}\alpha_1 e^{\sqrt{-1}\alpha_i y} p'_i(x) \\ -\sqrt{-1}\alpha_1 e^{-\sqrt{-1}\alpha_i y} \overline{p'_i(x)} \end{pmatrix}$$

Therefore we have

$$J\phi_*(\partial/\partial x) = (\phi_*(\partial/\partial x)) \times \phi(x, y)$$

is given by the following

$$(3.5) \quad \begin{aligned} J\phi_*(\partial/\partial x) &= \sqrt{-1}\varepsilon/2 \sum_{i=1}^3 (p'_i \overline{p_i} - p_i \overline{p'_i}) \\ &+ E_1 e^{\sqrt{-1}\alpha_1 y} (-\overline{p'_2(x) p_3(x)} + \overline{p_2(x) p'_3(x)}) \\ &+ E_2 e^{\sqrt{-1}\alpha_2 y} (-\overline{p'_3(x) p_1(x)} + \overline{p_1(x) p'_1(x)}) \\ &+ E_3 e^{\sqrt{-1}\alpha_3 y} (-\overline{p'_1(x) p_2(x)} + \overline{p_1(x) p'_2(x)}) \\ &+ \overline{E_1} e^{-\sqrt{-1}\alpha_1 y} (-p'_2(x) p_3(x) + p_2(x) p'_3(x)) \\ &+ \overline{E_2} e^{-\sqrt{-1}\alpha_2 y} (-p'_3(x) p_1(x) + p_3(x) p'_1(x)) \\ &+ \overline{E_3} e^{-\sqrt{-1}\alpha_3 y} (-p'_1(x) p_2(x) + p_1(x) p'_2(x)) \end{aligned}$$

From the above relations, the map  $(\mathbf{R}^2, \phi)$  is J-holomorphic curve of  $S^5$  if and only if

$$(3.6) \quad \sum_{i=1}^3 (p'_i \overline{p_i} - p_i \overline{p'_i}) = 0$$

$$(3.7) \quad \begin{pmatrix} 0 & -p_3 & p_2 \\ p_3 & 0 & -p_1 \\ -p_2 & p_1 & 0 \end{pmatrix} \begin{pmatrix} p'_1 \\ p'_2 \\ p'_3 \end{pmatrix} = -\sqrt{-1} \begin{pmatrix} \alpha_1 \overline{p_1} \\ \alpha_2 \overline{p_2} \\ \alpha_3 \overline{p_3} \end{pmatrix}$$

$$(3.8) \quad \sum_{i=1}^3 |p'_i(x)|^2 = \sum_{i=1}^3 \alpha_i^2 |p_i(x)|^2$$

The equations (3.7) reduce to the following form;

$$(3.9) \quad \begin{aligned} p'_1 &= \sqrt{-1}(\alpha_3 - \alpha_2)\overline{p_3 p_2} \\ p'_2 &= \sqrt{-1}(\alpha_1 - \alpha_3)\overline{p_1 p_3} \\ p'_3 &= \sqrt{-1}(\alpha_2 - \alpha_1)\overline{p_2 p_1}. \end{aligned}$$

The equations (3.9) imply the condition (3.6) automatically. Since  $\sum_{i=1}^3 |p_i(x)|^2 = 1$ , (3.8) and (3.9), we have also

$$(3.10) \quad \sum_{i=1}^3 \alpha_i |p_i(x)|^2 = 0,$$

Here, in particular if we assume that  $p_1(x), p_3(x)$  are real valued smooth functions and  $p_2(x) = \sqrt{-1}q_2(x)$ , for a certain real valued smooth function  $q_2(x)$ , then the equations (3.9), reduce to the Euler type equations.

$$(3.11) \quad \begin{aligned} p'_1 &= (\alpha_3 - \alpha_2)p_3 q_2 \\ q'_2 &= (\alpha_1 - \alpha_3)p_1 p_3 \\ p'_3 &= (\alpha_2 - \alpha_1)q_2 p_1. \end{aligned}$$

Therefore, by the well known fact about the Euler type equations, and hence the solution of (3.11) can be represented by using elliptic integrals, and hence there exist many required examples.

## 4. 3-dimensional submanifolds

### 4.1. Orbits

In this section, we give the examples of 3-dimensional totally real and CR-submanifolds of  $S^6$  which are obtained as the orbits of the subgroup  $Sp(1) \cong SU(2)$  of  $G_2$ . It was known that there exist 4-types of representations from  $SU(2)$  to  $G_2$  by Mal'cev([26]). We call these four types of the representations types I ~ IV, respectively. Mashimo([27]) proved that there exist 3-dimensional totally real (Lagrangian submanifold) corresponding to the types I ~ IV, respectively. In particular, typeIV

is a irreducible representation from  $SU(2)$  to  $G_2$ . The orbit of  $SU(2)$  gives the realization of the 3-dimensional totally real submanifold  $f : S^3(1/16) \rightarrow S^6$  which is obtained by Ejiri([10]). We note that any 3-dimensional totally real submanifold of  $S^6$  is a minimal submanifold.

We here recall the actions of  $SU(2) \subset G_2$  on  $\text{Im}\mathfrak{C}$  corresponding to the types I  $\sim$  IV, respectively. Action of type IF

$$\rho_{\text{I}}(q)(a + b\varepsilon) = a + (qb)\varepsilon.$$

Action of type III:

$$\rho_{\text{III}}(q)(a + b\varepsilon) = qa\bar{q} + (qb\bar{q})\varepsilon.$$

Action of type IV: The Lie algebra of  $SU(2)$  is spanned by the following basis (of (2.1)). We denote by  $U_4$  the Lie subgroup of  $G_2$  corresponding to the type IV.

$$\begin{cases} X_1 &= 4G_{32} + 2G_{54} + 6G_{76}, \\ X_2 &= \sqrt{6}(G_{37} + G_{26} - 2G_{15}) + \sqrt{10}(G_{42} - G_{35}), \\ X_3 &= \sqrt{6}(G_{63} + G_{27} - 2G_{41}) + \sqrt{10}(G_{25} - G_{34}). \end{cases}$$

Then  $U_4$  is isomorphic to  $SO(3)$ .

We shall consider the action of type II which includes some nice examples. We define the the action of type II ( $Sp(1)$  acts on  $\text{Im}\mathfrak{C}$ ) as follows;

$$\rho_{\text{II}}(q)(a + b\varepsilon) = qa\bar{q} + (b\bar{q})\varepsilon$$

where  $q \in S^3 \subset \mathbf{H}$ ,  $S^3 \cong Sp(1)$ . Then we can easily see that  $\{\rho_{\text{II}}(q)|q \in S^3\}$  is a Lie subgroup of  $G_2$ . In fact, we see that

$$\rho_{\text{II}}(q)(a + b\varepsilon)\rho_{\text{II}}(q)(c + d\varepsilon) = \rho_{\text{II}}(q)(ac - \bar{d}b + (da + b\bar{c})\varepsilon).$$

Next, we consider the orbit of  $Sp(1)$  through the point  $\sqrt{1 - r^2} + r\varepsilon$  of  $S^6$  given by

$$\psi_r(q) = \sqrt{1 - r^2}qi\bar{q} + r\bar{q}\varepsilon.$$

Then we see that the map  $\psi_r$  is an imbedding for for each  $r$  ( $0 \leq r \leq 1$ ). To obtain the 3-dimensional totally real or CR-submanifolds, the tangent space at  $\psi_r(q)$  is spanned by  $\{(\psi_r)_*(qi) = -r(i\bar{q})\varepsilon, (\psi_r)_*(qj) = -2\sqrt{1 - r^2}qk\bar{q} - r(j\bar{q})\varepsilon, (\psi_r)_*(qk) = 2\sqrt{1 - r^2}qj\bar{q} - r(k\bar{q})\varepsilon, \}$ . where  $\{qi, qj, qk\}$  is a left invariant vector field of  $S^3$ . Then the induced metric is given by

$$\begin{aligned} \langle (\psi_r)_*(qi), (\psi_r)_*(qi) \rangle &= r^2, \\ \langle (\psi_r)_*(qj), (\psi_r)_*(qj) \rangle &= \langle (\psi_r)_*(qk), (\psi_r)_*(qk) \rangle = 4 - 3r^2 \end{aligned}$$

the other elements are zero. In order to the Kähler angle  $\theta$  of the tangent space, we check the action of the almost complex structure on the tangent space;

$$J((\psi_r)_*(qi)) = (-r(i\bar{q})\varepsilon) \left( \sqrt{1 - r^2}qi\bar{q} + r\bar{q}\varepsilon \right)$$

$$\begin{aligned}
&= r \left( rqi\bar{q} - \sqrt{1-r^2}\bar{q}\varepsilon \right), \\
J((\psi_r)_*(qj)) &= (3r^2 - 2)qj\bar{q} - 3r\sqrt{1-r^2}(k\bar{q})\varepsilon, \\
J((\psi_r)_*(qk)) &= (3r^2 - 2)qk\bar{q} + 3r\sqrt{1-r^2}(j\bar{q})\varepsilon.
\end{aligned}$$

Since the vector field  $J((\psi_r)_*(qi))$  is normal to  $\psi_r(M)$ , the Kähler angle  $\theta$  is given by

$$\cos \theta = \left| \left\langle \frac{(\psi_r)_*(qj)}{|(\psi_r)_*(qj)|}, \frac{J((\psi_r)_*(qk))}{|(\psi_r)_*(qk)|} \right\rangle \right| = \frac{\sqrt{1-r^2}(9r^2-4)}{4-3r^2}.$$

By homogeneity, the Kähler angle of the orbit is constant. Also, we obtain

**Proposition 4.1.** (0) *If  $r = 0$ , then the orbit is a totally geodesic  $J$ -holomorphic curve.*

- (1) *If  $r = 2/3$ , then the orbit is a 3-dimensional totally real submanifold (which is not totally geodesic).*
- (2) *If  $r = 2\sqrt{2}/3$ , then the orbit is a 3-dimensional CR-submanifold.*
- (3) *If  $r = 1$ , then the orbit is a 3-dimensional totally geodesic, totally real submanifold*

We may note that the  $J$ -invariant 2-dimensional distribution of the above 3-dimensional CR-submanifold is not involutive.

For type I, each orbit is a 3-dimensional sphere in the quaternion, and if it is totally geodesic, then it is totally real. For type III, there exists an orbit whose image is a 3-dimensional totally real submanifold. There does not exist an orbit whose image is a 3-dimensional CR-submanifold. For type IV there exist two orbits whose image are 3-dimensional totally real submanifolds. There exist two parameter family of orbits whose image are 3-dimensional CR-submanifolds (see [27] and [20]).

## 4.2. Tubes

In this section, we shall give the construction of 3-dimensional  $\Sigma_J$  submanifolds from the image of the exponential map of some unit sphere bundle over  $J$ -holomorphic curves, we shall call such image "tube." It is well known that the exponential map at  $x \in S^6$ ,  $\exp_x : T_x(S^6) \rightarrow S^6$  with respect to the canonical metric is given by  $\exp_x(tX) = \cos t \cdot x + \sin t \cdot X$ . From the arguments in §3.2, we may observe that, if a  $J$ -holomorphic curve is not totally geodesic, the normal bundle can be represented by the direct sum of the 1st and the 2nd normal bundles. By this splitting, we may define the tubes of corresponding normal bundles as follows.

$$\begin{aligned}
\tilde{\varphi}_t^1(m, u) &= \cos t \cdot \varphi(m) + \sin t \cdot \sigma(u, u)/\lambda, \\
\tilde{\varphi}_t^2(m, u) &= \cos t \cdot \varphi(m) + \sin t \cdot \varphi_*(u) \times \sigma(u, u)/\lambda,
\end{aligned}$$

where  $\varphi : M \rightarrow S^6$  is a  $J$ -holomorphic curve without geodesic point,  $(m, u)$  is an element of the unit tangent bundle of  $M$ . Also we note that  $\lambda = |\sigma(u, u)|$  is depend only on the point  $m \in M$ . Then we have

**Proposition 4.2.** ([8, 21]) *The tubes of second normal bundles imply the following.*

- (0) *If  $\cos t = 0$ , then it is the given  $J$ -holomorphic curve.*
- (1) *If  $\cos t = 2/3$ , then it is a 3-dimensional totally real submanifold.*
- (2) *If  $\cos t = 2\sqrt{2}/3$ , then it is a 3-dimensional CR-submanifold.*
- (3) *If  $\cos t = 1$ , then it is a 3-dimensional totally real submanifold.*

We can obtain the examples of submanifold with constant Kähler angle as the tubes of the 1st normal bundle, but it is more complicated ([21]). By Theorem 3.4 and Proposition 4.2, we can easily see that there exists a (non-trivial) deformation of 3-dimensional totally real or CR-submanifolds. We remark that the values of Propositions 4.1, 4.2 coincide. However we can not understand the reason up to now.

### 4.3. Flows

We introduce the another construction of the  $J$ -holomorphic curves and 3-dimensional submanifolds. A 3-dimensional CR-submanifold is called *involutive type* if the 2-dimensional  $J$ -invariant distribution is involutive. Such a 3-dimensional CR-submanifold was obtained by the second author ([33]). The immersion

$$\varphi : S^1 \times S^2 \rightarrow S^5 \subset S^6$$

is given by

$$\varphi(t, p) = \exp(tX)(p)$$

where  $X = aG_{51} + bG_{62} + cG_{73}$  ( $a+b+c = 0, abc \neq 0$ ) and  $p \in S^2$  and  $\exp$  is a exponential map of the matrix. The induced metric of this example is warped product, and its cohomogeneity is 1 (not homogeneous with respect to the induced metric). These examples are extended by ([20]). In ([23]), we classify any  $J$ -holomorphic curves from 2-dimensional torus to  $S^5$  (which is a totally geodesic sphere of  $S^6$ ) by using the integrable systems. Explicit examples of such curves is given by

$$\varphi(t, s) = \exp(tX)(\gamma(s))$$

where  $\gamma(s)$  is some curve of  $S^2$  which is represented by the elliptic functions. In particular, if the curve  $\gamma(s)$  is a geodesic in  $S^2$ , then the immersion coincide with flat torus. From these construction, if we take the irrational flow, we get the non-compact examples.

Concerning 4-dimensional CR-submanifolds of  $S^6$ , it is only known that there does not exist a 4-dimensional CR-product submanifold of  $S^6$  ([33]).

## 5. 4-dimensional CR-submanifolds

Concerning 4-dimensional CR-submanifolds of  $S^6$ , it is only known that there does not exist a 4-dimensional CR-product submanifold of  $S^6$  ([33]).

### 5.1. Fundamental properties

First, we recall the following characterization for a 4-dimensional oriented submanifold of  $S^6$  to be a CR-submanifolds. ([8, 21])

**Proposition 5.1.** (5.1) *Let  $\varphi : M^4 \rightarrow S^6$  be an orientable 4-dimensional submanifold of  $S^6$ . Then  $(M^4, \varphi)$  is a CR-submanifold of  $S^6$  if and only if it satisfies one of the following conditions*

- (1)  $\omega(T^\perp M^4) = 0$ ,
- (2)  $*\omega(TM^4) = 0$ ,

where  $\omega$  denote the Kähler form of  $S^6$ .

**Remark 5.1.** Let  $\varphi : M^4 \rightarrow S^6$  be a 4-dimensional CR-submanifold. If  $g \in G_2$ , then  $g \circ \varphi$  is also. However, if  $h \in SO(7) \setminus G_2$ , then  $h \circ \varphi$  is not a CR-submanifold, in general.

Let  $\varphi : M^4 \rightarrow S^6$  be a 4-dimensional CR-submanifolds of  $S^6$  and discuss some fundamental properties concerning  $(M^4, \varphi)$ . Especially we discuss a (local) orthonormal CR-adapted frame field along  $(M^4, \varphi)$ . Let  $\xi_1, \xi_2$  be a local orthonormal frame fields  $\xi_1, \xi_2$  of  $H^\perp$ . Then we have  $\text{span}_R\{J\xi_1, J\xi_2\} = T^\perp M^4$ . Then the exterior product  $\xi_1 \times \xi_2$  depends only on the given orientation of  $H^\perp$  and is independent on the choice of the orthonormal frame fields. Also we have  $\xi_1 \times \xi_2, J(\xi_1 \times \xi_2) \in H$ . Therefore, the vector field  $\xi_1 \times \xi_2$  is well defined whole on  $M^4$ . Hence  $H$  has an absolute parallelisability. We see that  $\{\xi_1 \times \xi_2, J(\xi_1 \times \xi_2), \xi_1, \xi_2\}$  is a local orthonormal frame field of  $M^4$ . We obtain

**Proposition 5.2.** *Let  $\varphi : M^4 \rightarrow S^6$  be an orientable compact 4-dimensional CR-submanifold of  $S^6$ , then the Euler number of  $M^4$  vanishes.*

By Proposition 5.2, we may immediately see that 4-dimensional sphere, product of two 2-dimensional spheres and complex 2-dimensional projective space can not be realized as a CR-submanifold of  $S^6$ . On the other hand, since  $\dim H^\perp = 2$ , and

$H^\perp$  is orientable, we can define two kinds of almost complex structures  $J_1, J_2$  on  $M^4$  such that

$$J_1 = J_H \oplus J', J_2 = J_H \oplus (-J')$$

where  $J_H$  is the restriction of the almost complex structure of  $S^6$  to the holomorphic distribution  $H$ . Hence we have the following decomposition

$$TS^6|_{\varphi(M^4)} = H \oplus H^\perp \oplus T^\perp M^4.$$

If we set  $V = H^\perp \oplus T^\perp M^4$ , then  $V$  is a  $\mathbf{C}^2$ -vector bundle over  $M^4$ . Concerning the characteristic classes of these vector bundles, we have the following

**Proposition 5.3.** (1)  $e(H) = c_1(H^{(1,0)}) = 0$ ,

$$(2) p_1(TM^4) = \{c_1(H^{(1,0)})\}^2 = -\{c_1(T^{(1,0)}M^4)\}^2,$$

$$(3) p_1(V) = 0,$$

$$(4) c_1(V^{(1,0)}) = 0,$$

in  $H^*(M, \mathbf{Z})$ , where  $p_1(*)$  (resp.  $c_1(*)$ ) is the first Pontrjagin (resp. Chern) class of the corresponding vector bundle, and  $e(*)$  is the Euler class.

**Corollary 5.1.** *If the holomorphic distribution  $H$  is involutive then each compact leaf of  $H$  is homeomorphic to 2-dimensional torus.*

Further, we have

**Theorem 5.1.** *Let  $\varphi : M^4 \rightarrow S^6$  be a 4-dimensional CR-submanifold of  $S^6$ . The first Pontrjagin class of the tangent bundle vanishes i.e.,  $p_1(TM^4) = 0$ .*

By taking account of the  $G_2$ -structure equations on  $S^6$ , we can also show that 2-dimensional totally real distribution  $H^\perp$  is not integrable.

## 5.2. Examples

The above arguments in §5.1 assert that there exist many obstructions for the existence of 4-dimensional CR-submanifolds of  $S^6$ . However, contrary to this circumstances, we may construct several examples of such submanifolds. We herewith introduce two typical examples of 4-dimensional CR-submanifolds of  $S^6$ .

**Example 1.** Let  $\gamma : I \rightarrow S^2 \subset \text{Im}\mathbf{H}$  be any curve in the 2-dimensional sphere  $S^2 \subset \text{Im}\mathbf{H} \simeq \mathbf{R}^3$ , and  $(q \in) S^3 \subset \mathbf{H}$  be the 3-dimensional sphere of the quaternions  $\mathbf{H}$ . Then the product immersion  $\psi : I \times S^3 \rightarrow S^6$ , which is defined by

$$\psi(t, q) = a\gamma(t) + b\bar{q}\varepsilon$$

gives a 4-dimensional submanifold of  $S^6$ , for any  $a, b > 0$  with  $a^2 + b^2 = 1$ . Here,  $t$  denotes the arc length parameter of  $\gamma$ .

In fact, let  $(I \times S^3, \psi)$  be the submanifold of  $S^6$  gives in the above example 1. Then, we may choose the orthonormal frame field  $\{\nu_1, \nu_2\}$  of the normal bundle in such a way that

$$\nu_1 = \dot{\gamma}(t) \times \gamma(t), \nu_2 = b\gamma(t) - a\bar{q}\varepsilon.$$

Thus we have

$$J(\nu_2) = (b\gamma(t) - a\bar{q}\varepsilon) \times (a\gamma(t) + b\bar{q}\varepsilon) = \gamma(t) \times \bar{q}\varepsilon = (\bar{q}\gamma(t))\varepsilon \in \mathbf{H}\varepsilon,$$

therefore,  $\langle \nu_1, J(\nu_2) \rangle = 0$ . Thus (1) of Proposition 5.1, we get the desired result. Further, we may obtain the corresponding CR-frame fields along  $(I \times S^3, \psi)$  in the following way. A local orthonormal frame field of  $H^\perp$  is given by

$$\begin{aligned} \psi_*(\xi_1) &= J(\nu_1) = \nu_1 \times \psi = -a\dot{\gamma}(t) + b(\dot{\gamma}(t) \times \gamma(t)) \cdot \bar{q}\varepsilon, \\ \psi_*(\xi_2) &= J(\nu_2) = \gamma(t) \times \bar{q}\varepsilon. \end{aligned}$$

On the other hand, an orthonormal frame field of  $H$  is given by

$$\psi_*(e_1) = J(\nu_1) \times J(\nu_2) = b\dot{\gamma}(t) + a(\dot{\gamma}(t) \times \gamma(t)) \cdot \bar{q}\varepsilon, \psi_*(J(e_1)) = (\dot{\gamma}(t)) \cdot \bar{q}\varepsilon.$$

**Example 2.** The following immersion  $\phi : S^1 \times S^3 \rightarrow S^6$  is a 4-dimensional CR-submanifold of  $S^6$ ;

$$\phi(\theta, q) = a(qi\bar{q}) + b(\tau(\theta)\bar{q}) \cdot \varepsilon,$$

for  $a, b > 0$  with  $a^2 + b^2 = 1$ , where  $\tau(\theta) = t\{-\sin(\theta) + \cos(\theta)i\} + s\{\cos(\theta)j + \sin(\theta)k\}$  is a great circle of  $S^3 \subset H$  for  $t, s > 0$  with  $t^2 + s^2 = 1$ .

**Remark 5.2.** The parameters  $(a, t) \in (0, 1) \times (0, 1)$  in the above Example 2 represent a parameterization for the orbit space consisting of the families of some  $U(2)$ -orbits through the points of  $S^6$ .

In fact, the basis of the tangent space of  $\phi$  is given by

$$\begin{aligned} \phi_*\left(\frac{\partial}{\partial \theta}\right) &= b(\tau'(\theta)\bar{q}) \cdot \varepsilon \\ \phi_*(qi) &= -b(\tau(\theta)i\bar{q}) \cdot \varepsilon \\ \phi_*(qj) &= -2a(qk\bar{q}) - b(\tau(\theta)j\bar{q}) \cdot \varepsilon \\ \phi_*(qk) &= 2a(qj\bar{q}) - b(\tau(\theta)k\bar{q}) \cdot \varepsilon. \end{aligned}$$

The following normal vector fields  $\{\nu_1, \nu_2\}$  are given by

$$\begin{aligned} \nu_1 &= b(qi\bar{q}) - a(\tau(\theta)\bar{q}) \cdot \varepsilon \\ \nu_2 &= \frac{1}{\sqrt{1+3a^2}} \left( b(qj\bar{q}) + 2a(\tau(\theta)k\bar{q}) \cdot \varepsilon \right), \end{aligned}$$

give rise to the orthonormal frame of  $T^\perp M^4$ . Therefore  $J(\nu_1) = \left( \tau(\theta)i\bar{q} \right) \cdot \varepsilon$ , and we have  $\langle \nu_1, J(\nu_2) \rangle = 0$ . Hence, by (1) of Proposition 5.1 we see that  $(S^1 \times S^3, \phi)$  a 4-dimensional CR-submanifold of  $S^6$ . We shall state the geometrical properties of the above examples.

We see that the Example 1 include the special one obtained by the orbit of the subgroup of  $G_2$  determined by the following representation  $\tau_I$  from  $U(2)$  to  $G_2$ ;

$$\tau_I(\theta, q)(u + v\varepsilon) = e^{i\theta}ue^{-i\theta} + (qve^{-i\theta})\varepsilon.$$

We see also that the Example 2 include the special one obtained by the orbit of the subgroup of  $G_2$  determined by the following representation  $\tau_{II}$  of  $U(2)$  of  $G_2$ ;

$$\tau_{II}(\theta, q)(u + v\varepsilon) = qu\bar{q} + (e^{i\theta}v\bar{q})\varepsilon.$$

Here  $e^{i\theta} = \cos \theta + (\sin \theta)i$  and  $i$  is the unit element of the quaternion. We note that the above two examples of 4-dimensional CR-submanifolds of  $S^6$  can be considered as the total space of  $S^1$ -bundle over  $S^1 \times S^2$  and  $S^2 \times S^1$ , respectively.

**Proposition 5.4.** *Let  $\phi : S^1 \times S^3 \rightarrow S^6$  be a 4-dimensional CR-submanifold of  $S^6$  in Example 2. Then*

- (1) *The map  $\phi$  is not an imbedding. In fact, we have  $\phi(\theta + \pi, -q) = \phi(\theta, q)$ . The immersion  $\phi$  is full.*
- (2) *The immersion  $\phi : S^1 \times S^3 \rightarrow S^6$  is minimal if and only if  $a = \sqrt{(3 + \sqrt{57})/24}$ ,  $t = 1/\sqrt{2}$ . For the other  $(a, t)$  in example 2, the length of the mean curvature vector field is constant, but the mean curvature vector field is not parallel with respect to the normal connection. In particular, the second fundamental is not parallel for any immersion of this type.*
- (3) *The normal curvature of the immersion  $\phi$  is not flat.*
- (4) *The Ricci eigenvalues of the induced metric of the immersion  $\psi_1$  are constant, but the metric is not Einstein.*
- (5) *If  $a = 1/\sqrt{3}$  and  $t = 1/\sqrt{2}$ , the holomorphic distribution  $H$  is integrable.*

**Remark 5.3.** We here describe the CR-frame field on  $S^1 \times S^3$  of the immersion  $\phi$ , explicitly;

$$\begin{aligned} \xi_1 &= \frac{1}{b}qi, \\ \xi_2 &= \frac{-1}{\sqrt{1+3a^2}} \left\{ \frac{5-9a^2}{4bst} \left( \frac{\partial}{\partial \theta} + (t^2 - s^2)qi \right) + \frac{3b}{2}qj \right\}, \end{aligned}$$

$$\begin{aligned}\tilde{e}_1 &= \frac{1}{\sqrt{1+3a^2}} \left\{ \frac{1-9a^2}{4ast} \left( \frac{\partial}{\partial \theta} + (t^2 - s^2)qi \right) + \frac{1-3a^2}{2a} qj \right\}, \\ J\tilde{e}_1 &= -\frac{1}{\sqrt{1+3a^2}} qk.\end{aligned}$$

The image of the CR-frame field by the differential of  $\phi$  is given by

$$\begin{aligned}\phi_*(\xi_1) &= -(\tau(\theta)i\bar{q})\varepsilon (= -J\nu_1), \\ \phi_*(\xi_2) &= \frac{-1}{\sqrt{1+3a^2}} \{-3ab \cdot qk\bar{q} + (1-3a^2)(\tau(\theta)j\bar{q})\varepsilon\} (= -J\nu_2), \\ \phi_*(\tilde{e}_1) &= \frac{1}{\sqrt{1+3a^2}} \{(3a^2-1) \cdot qk\bar{q} - 3ab(\tau(\theta)j\bar{q})\varepsilon\} (= J\nu_1 \times J\nu_2), \\ J\phi_*(\tilde{e}_1) &= \frac{1}{\sqrt{1+3a^2}} \{2a \cdot qj\bar{q} - b(\tau(\theta)k\bar{q})\varepsilon\}.\end{aligned}$$

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